# Proof by contradiction

- direct proof $P \Rightarrow Q$
  Assume P true, show Q follows
- contrapositive: Assume Q false, show P false follows
  $(\neg Q \Rightarrow \neg P)$
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**Proposition:** The square root of 2 is not a rational number.

~~A rational number~~ $n$ is a rational number if there are integers $x, y$, with $y \neq 0$ so that $n = \frac{x}{y}$.

If $\sqrt{2}$ were rational $\sqrt{2} = \frac{x}{y}$

$y\sqrt{2} = x$

$2y^2 = x^2$

$x^2 - 2y^2 = 0.$

(Diophantine Eq'n)

$\sqrt{2}$ rational $\Leftrightarrow$ there are integers $x, y, y$ so that $y \neq 0$, $xy \in \mathbb{Z}$.

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$x^2 + y^2 = z^2$

Find $x, y, z$ solving this

**Lemma:** If $a^2$ is even, then $a$ is even.

**Proof:** We will prove the contrapositive, which says that if $a$ is odd, then $a^2$ is odd. Suppose $a$ is odd. Then $a = 2k + 1$ for some $k$. Therefore $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ is odd.

**Proof of Proposition:** Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers $a$ and $b$ with $b \neq 0$ so that

$$2 = (\frac{a}{b})^2.$$

**Proof of Proposition:** Suppose that $\sqrt{2}$ is a rational number. Then we can find positive integers $a$ and $b$ with $b \neq 0$ so that

$$a^2 - 2b^2 = 0.$$

This means $a^2 = 2b^2$ so $a^2$ is even.

So $a$ is even. $a = 2a_1$

$$(2a_1)^2 - 2b^2 = 0$$

$$4a_1^2 - 2b^2 = 0$$

$$2a_1^2 - b^2 = 0$$

So $b^2$ is even, so $b$ is even

Suppose $\sqrt{2} = \left(\frac{a}{b}\right)$ where $a, b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a,b) = 1$.

Then $a^2 - 2b^2 = 0$

as we saw above this means $a$ and $b$ are both even.

Conclude $a$ and $b$ can't exist as described

# Logical Structure of proof by contradiction

- A contradiction is a statement of the form ($C$ and $\sim C$) which is *always false.*
- The strategy of proof by contradiction is that if $A \implies B$ is true, and $B$ is false, then $A$ is false.

**Proposition:** $P \implies Q$.

- Assume $P$ is true.
- Assume ($P$ and $\sim Q$) is true and

$$P \text{ and } \sim Q \implies C \text{ and } \sim C$$

  for some statement $C$.
- If ($P$ and $\sim Q$) implies ($C$ and $\sim C$) is a true implication yielding a false conclusion, then the hypothesis must be false.
- Therefore ($P$ and $\sim Q$) is false.
- If ($P$ and $\sim Q$) is false, and $P$ is true then $\sim Q$ is false
- $Q$ is true.